Moriah College

בית ספר הר המוריה

2021 **HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

Mathematics Advanced

General Instructions

Reading time – 10 minutes

- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks:100

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–25)

- Attempt Questions 11–30
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Answer clearly on your own paper

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1 A function is given by $f(x)=\sqrt{4-x^2}$. What is the domain of this function?

A. $-2<x<2$

B. $x\le 2$

C. $-4\le x\le 4$

D. $-2\le x\le 2$

2 Which would be the best description of $h(x)=x^4+1$

A. one to one

B. many to one

C. one to many

D. many to many

3 What is $\int(\pi+\cos(\pi x))\,dx$?

A. $\pi x+\sin(\pi x)+C$

B. $\pi x+\dfrac{\sin(\pi x)}{\pi}+C$

C. $\pi+\dfrac{\sin(\pi x)}{\pi}+C$

D. $\dfrac{\pi^2}{2}+\sin(\pi x)+C$

4 Which interval gives the range of the relation $x^2 + (y-1)^2 = 4$?

A. $[-2, 2]$

B. $[-2, 2]$

C. $[0,1]$

D. $[-1, 3]$

5 A box contains six red marbles and four blue marbles. Two marbles are drawn from the box without replacement. What is the probability they are both the same colour?

A. $\frac{1}{2}$

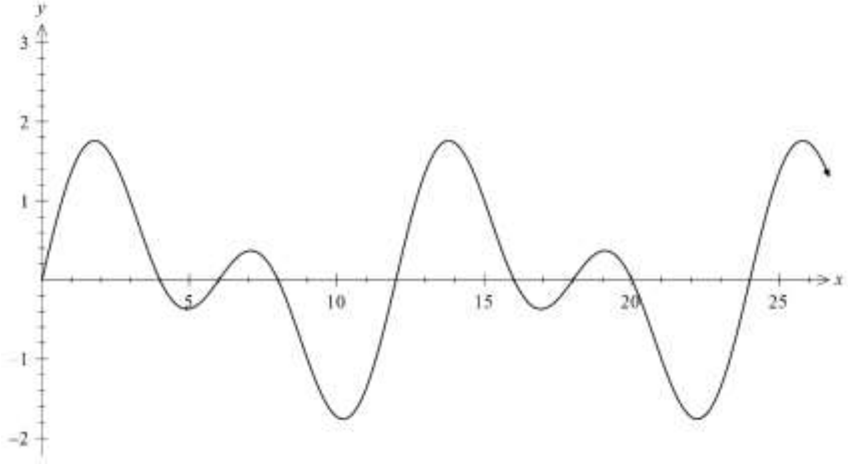
B. $\frac{28}{45}$

C. $\frac{7}{15}$

D. $\frac{3}{5}$

6 Part of the graph of the function $y = \sin\left(\frac{\pi x}{3}\right) + \sin\left(\frac{\pi x}{6}\right)$, where $x \geq 0$ is shown below.

What is the period of the function?



A. 6

B. 10

C. 12

D. 24

7 It is given that $2\ln x = 3\ln y + 1$, where x and y are both positive.
Which of the following statements is true?

A. $2x - 3y = e$

B. $x^2 = y^3 + e$

C. $\frac{x^2}{y^3} = 1$

D. $\frac{x^2}{y^3} = e$

8 In an exam worth 100 marks, Billie got 75 and was 2 standard deviations above the mean. Charlie got 45 and was one standard deviation below the mean. If this a normal distribution, what is the probability of Denise achieving a score of 85 or more?

A. 0.05

B. 0.025

C. 0.34

D. 0.0015

9. The function $y = f(x)$ has a stationary point at (2,1).

Which one of the following is definitely a stationary point of $y = -f\left(\frac{x}{3}\right) + 4$?

A. $\left(\frac{2}{3}, -3\right)$

B. $\left(\frac{2}{3}, 3\right)$

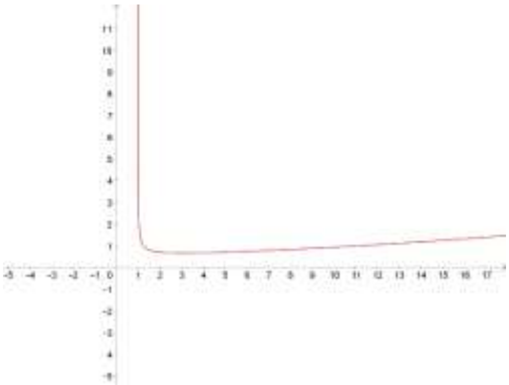
C. $(6, 3)$

D. $(6, -3)$

10. Two functions are defined such that $f(x) = 2^x$ and $g(x) = \sqrt{x-1}$.

Both functions are defined over their largest possible domains. If $h(x) = f(g(x))$, which of the following represents $y = h'(x)$?

A.



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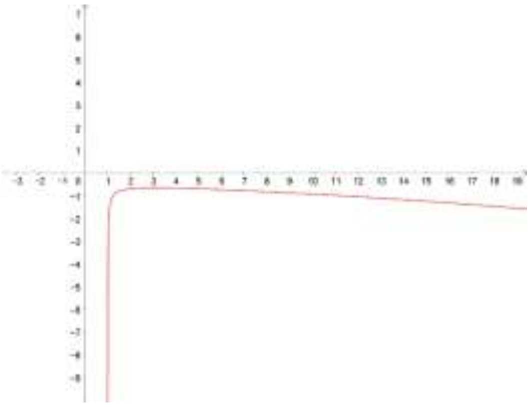

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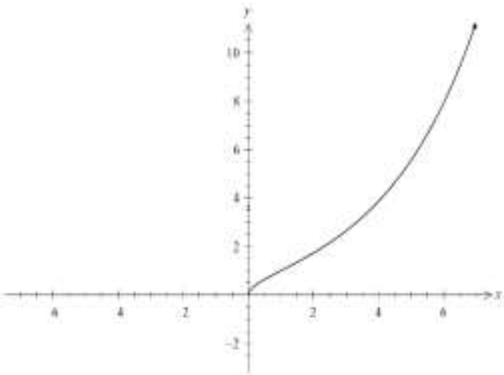

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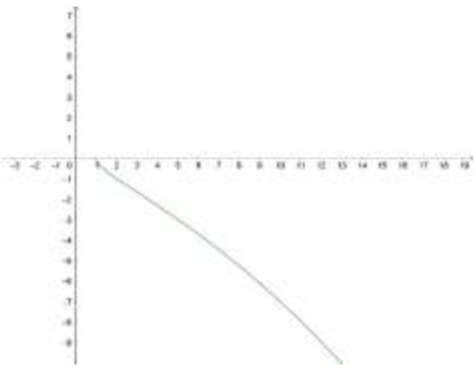

Mathematics Advanced Section II

Answer Booklet 1 Section II

90 marks
Attempt Questions 11–30

Allow about 2 hours and 45 minutes for this section.
Section II

Question 11 (3 marks)

The following table shows a probability distribution of a discrete random variable *x*.

X	**0**	**1**	**2**	**3**
$P(X = x)$	$\frac{3}{13}$	$\frac{5}{13}$	$\frac{2}{13}$	k

(a) Find the value of *k*. **1**

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(b) Find E(*X*). **2**

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Question 12 (2 marks)

A crystal growth furnace is designed to manufacture components for a space shuttle. **2**
For proper growth the temperature must by controlled accurately by adjusting the input power.
Suppose the relationship is given by $T = 0.1w^2 + 2.155w + 20$,
where T is the temperature in Celsius and w is the (positive) input power in watts.
How much power is needed to maintain the temperature at $200°C$?

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Question 13 (4 marks)

For the arithmetic sequence 10, 4, -2 ….

(a) Calculate the value of the fourteenth term. **2**

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(b) Find the sum of the first fourteen terms. **2**

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Question 14 (2 marks)

Evaluate $\int_0^4 x\left(x^2+6\right)dx$ **2**

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Question 15 (5 marks)

Nicky has opened a business to customise skateboards.

The cost of production is given by $C = 20x + 600$ dollars and income earned is given by $I = 50x$, where x is the number of skateboards produced.

(a) If the production cost is $800, how many skateboards have been manufactured? **1**

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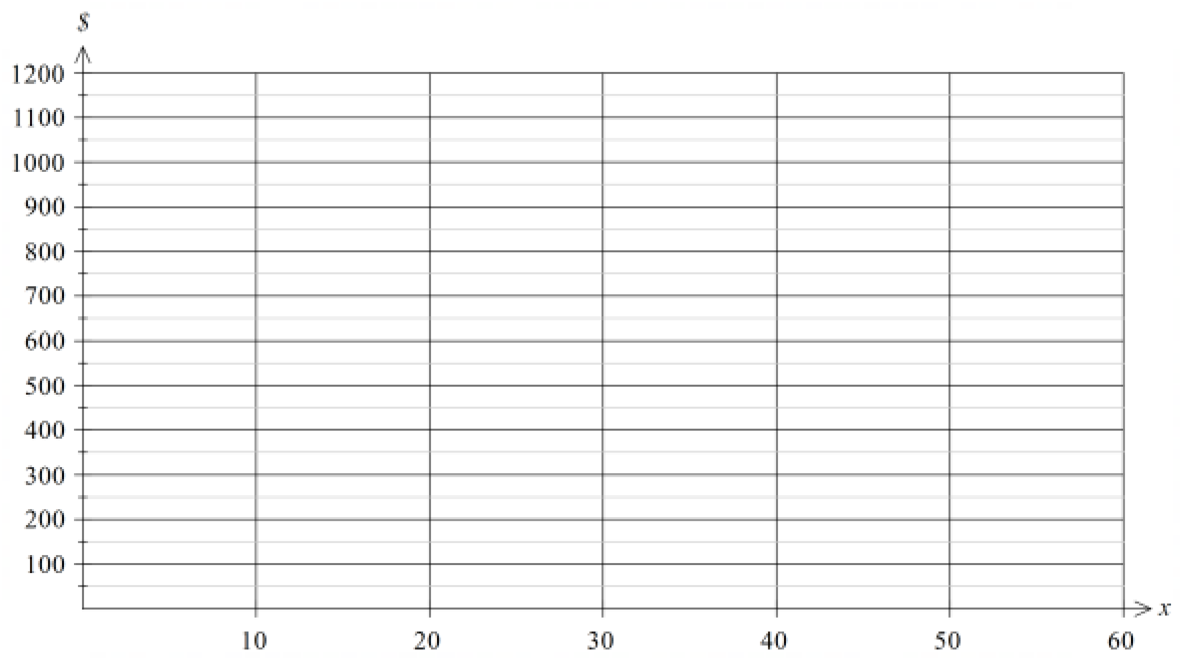
(b) Graph both equations on the axes provided. **2**



Question 15 (continued)

(c) What is the minimum number of skateboards that Nicky must sell to make a profit? **2**

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Question 16 (3 marks)

Given that $\cos\alpha = \frac{5}{6}$, and that $\tan\alpha < 0$ find the exact value of $\sin\alpha \times \sec\alpha$. **3**

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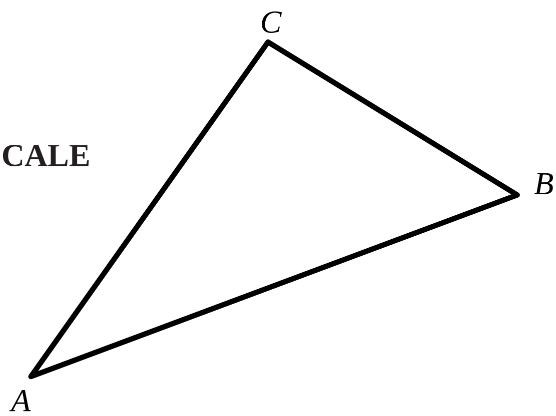
Question 17 (3 marks)

A runner leaves a point A and runs for 4 km on a bearing of $N15°E$.
She then runs 2 km SW from C to B.

(a) Use the diagram below to show why $\angle ACB = 60°$. **1**

NOT TO SCALE



(b) Use the cosine rule to determine the exact distance from A to B. **2**

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Question 18 (3 marks)

In the city of Tangentia the average number of books in a residence is normally distributed with a mean of 10 books and a variance of 9 books. If two households are chosen at random Show that the probability at least one of them has between 4 and 13 books is 0.97 (2 dp) **3**

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Question 19 (5 marks)

(a) Use the fact that $\sec x = (\cos x)^{-1}$ to show that $\frac{d}{dx}(\sec x) = \sec x \tan x$. **2**

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(b) Hence find $\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \, dx$ **3**

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Question 20 (6 marks)

(a) A function is defined by $f(x) = 2^{-x} + 1$. Complete the table of values below. **1**

x	0	1	2
$f(x)$			

(b) Use the trapezoidal rule with all the above function values to estimate the value of $\int_0^2 \left(2^{-x} + 1\right) dx$ correct to three decimal places. **2**

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(c) Find the value of $\int_0^2 \left(2^{-x} + 1\right) dx$ correct to three decimal places. **2**

(d) Why would you expect the value of the integral to be less than the approximation? (A diagram may prove useful.) **1**

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Question 21 (8 marks)

A curve is defined by $y = x^3 - 3x^2 + 4$ over the domain $-2 \le x \le 3$.

(a) Find any turning points, identifying their nature. **3**

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(b) Find any points of inflection. **2**

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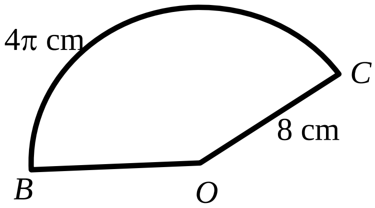
Question 21 (continued)

(c) Draw a neat sketch of $y = x^3 - 3x^2 + 4$ over the domain $-1.5 \leq x \leq 3$, given the curve passes through (-1,0). Show intercepts, turning points, inflection(s) and endpoints. **3**

Question 22 (2 marks)



NOT TO SCALE

OBC is a sector of a circle centre O and radius 8cm. The length of the arc *BC* is 4π cm. **2**
Calculate the exact area of the sector *OBC*.

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Question 23 (3 marks)

The time taken to complete a building project varies inversely with the number of builders. **3**
If the building can be completed in 180 days by 55 workers, find how many workers will be needed to finish the project in exactly 99 days

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Question 24 (3 marks)

Solve $\ln(8-2x)=\frac{1}{2}\ln(4-x)$. **3**

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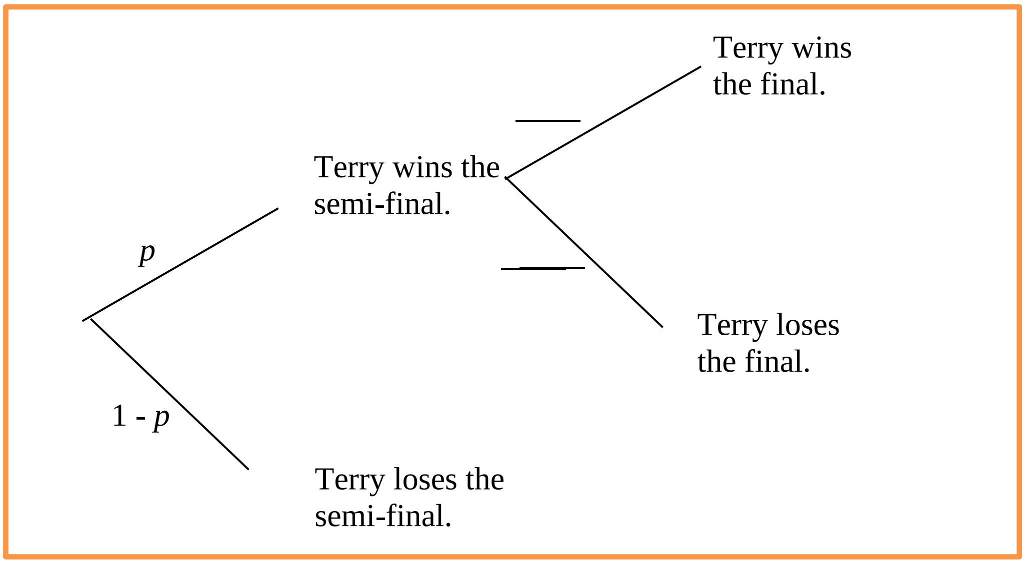
Question 25 (5 marks)

Terry will play in the semi-final of a tennis tournament.
If Terry wins the semi-final she will progress to the final. If she loses she will not progress.
If Terry wins the final she will be champion.
The probability that Terry will win the semi-final is p.
If she wins the semi-final then the probability she will be the champion is 0.6.



(a) Complete the values in the tree diagram. **1**

(b) The probability that Terry will not be the champion is 0.58. Find the value of p. **2**

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(c) Given that Terry did not become the champion, find the probability she lost in the semi-final. **2**

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Question 26 (5 marks)

The half-life of a substance is the time taken to decay to half the original mass.
Strontium-90 has a half-life of 28 days. Its mass S is given by $S = S_0 e^{-kt}$ where S is measured in mg and t is measured in days. A sample has an initial mass of 50 mg.

(a) Find the value of k correct to 3 decimal places. **2**

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(b) Use the exact value of k to find the mass remaining after 40 days. **1**

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(c) How long does it take (to the nearest day) for the sample to decay to a mass of 2 mg? **2**

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Question 27 (7 marks)

Two functions are given by $y = 4\cos\left(\frac{\pi}{16}x\right)$ and $y = -\frac{1}{2}x + 4$

(a) Show that the functions have a point of intersection at (8,0) **2**

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(b) Sketch both functions over the domain [0,8] **2**

(c) Find the exact area enclosed by the curves over this domain. **3**

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Question 28 (9 marks)

(a) If $y = 1 - xe^{-x}$ find $\frac{dy}{dx}$. **3**

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(b) Find the equations of the tangent and the normal to $y = 1 - xe^{-x}$ at the point $P(0,1)$ **4**

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Question 28 (continued)

(c) If the tangent and normal meet the *x*-axis at *A* and *B* respectively, find the area of the triangle *ABP*. **2**

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Question 29 (6 marks)

Albert borrows \$200,000 at 6% per annum compounded monthly. He makes monthly payments of \$2000 at the end of each month. B_n is the balance owing after *n* months,

(a) Show that the amount owing after 2 years is given by $B_{24} = \$174568.04$ **3**

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Question 29 (continued)

(b) Albert decides he wants to pay the balance of the loan off after a further 8 years. What should his new monthly payment be? **3**

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Question 30 (6 marks)

A probability density function is given by $f(x)=\begin{cases}\dfrac{1}{2\sqrt{4-x}} & ;0\le x\le 3\\ 0 & ;\text{otherwise}\end{cases}$

(a) Prove that this is a valid probability density function, (you may assume $f(x)\ge 0$). **2**

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(b) Find the cumulative distribution function. **2**

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(c) Find the median of the probability density function. **2**

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END OF EXAMINATION

Teacher Training Australia

2021 **HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

Mathematics Advanced SOLUTIONS

General Instructions

Reading time – 10 minutes

- Working time – 3 hours
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Total marks:100

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–25)

- Attempt Questions 11–30
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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 A function is given by $f(x)=\sqrt{4-x^2}$. What is the domain of this function?

A. $-2 < x < 2$

B. $x \leq 2$

C. $-4 \leq x \leq 4$

D. $-2 \leq x \leq 2$

2 Which would be the best description of $h(x)=x^4+1$

A. one to one

B. many to one

C. one to many

D. many to many

3 What is $\int\left(\pi+\cos(\pi x)\right)dx$?

A. $\pi x+\sin(\pi x)+C$

B. $\pi x+\dfrac{\sin(\pi x)}{\pi}+C$

C. $\pi+\dfrac{\sin(\pi x)}{\pi}+C$

D. $\dfrac{\pi^2}{2}+\sin(\pi x)+C$

4 Which interval gives the range of the relation $x^2 + (y-1)^2 = 4$?

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C. $[0,1]$

D. $[-1, 3]$

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A. $\frac{1}{2}$

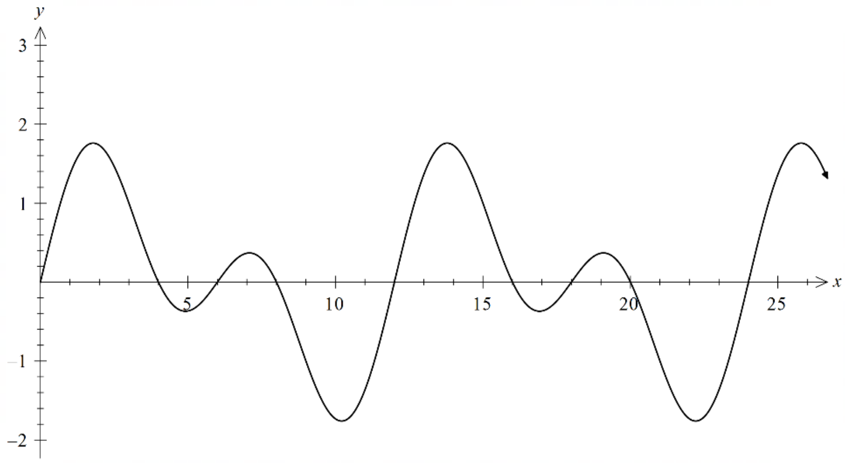
B. $\frac{28}{45}$

C. $\frac{7}{15}$

D. $\frac{3}{5}$

6 Part of the graph of the function $y = \sin\left(\frac{\pi x}{3}\right) + \sin\left(\frac{\pi x}{6}\right)$, where $x \geq 0$ is shown below.

What is the period of the function?



A. 6

B. 10

C. 12

D. 24

7 It is given that $2\ln x = 3\ln y + 1,$ where x and y are both positive.
Which of the following statements is true?

A. $2x - 3y = e$

B. $x^2 = y^3 + e$

C. $\frac{x^2}{y^3} = 1$

D. $\frac{x^2}{y^3} = e$

8 In an exam worth 100 marks, Billie got 75 and was 2 standard deviations above the mean. Charlie got 45 and was one standard deviation below the mean. If this a normal distribution, what is the probability of Denise achieving a score of 8.5 or more?

A. 0.05

B. 0.025

C. 0.34

D. 0.0015

9 The function $y = f(x)$ has a stationary point at (2,1).

Which one of the following is definitely a stationary point of $y = -f\left(\frac{x}{3}\right) + 4$?

A. $\left(\frac{2}{3}, -3\right)$

B. $\left(\frac{2}{3}, 3\right)$

C. $(6, 3)$

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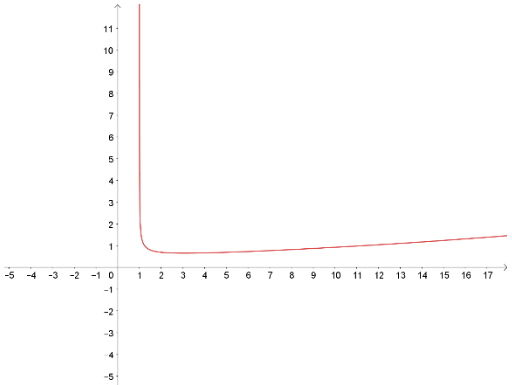
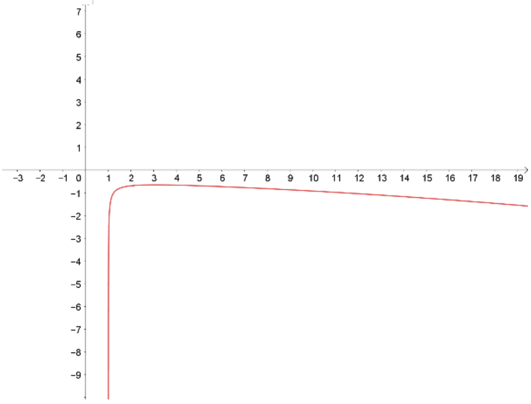
10 Two functions are defined such that $f(x) = 2^x$ and $g(x) = \sqrt{x-1}$.

Both functions are defined over their largest possible domains. If $h(x) = f(g(x))$, which of the following represents $y = h'(x)$?

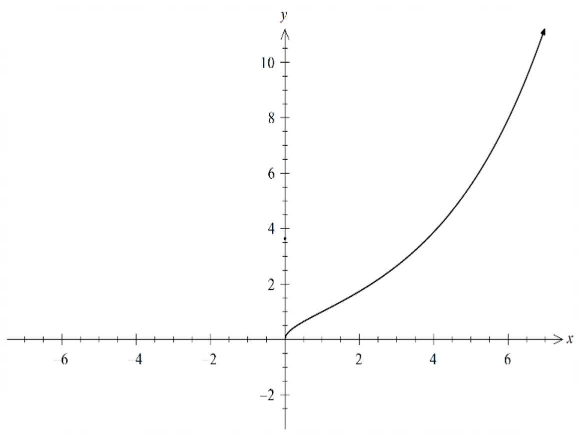
A.



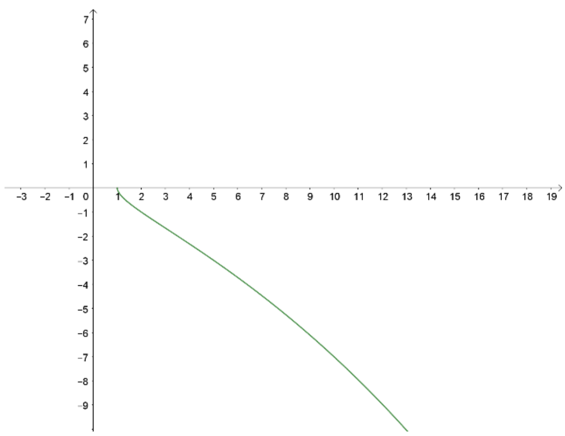
B.



C.



D.



Mathematics Advanced Section II

Answer Booklet 1 Section II

90 marks
Attempt Questions 11–30
Allow about 2 hours and 45 minutes for this section.

Section II

Question 11 (3 marks)

The following table shows a probability distribution of a discrete random variable *x*.

X	**0**	**1**	**2**	**3**
$P(X = x)$	$\frac{3}{13}$	$\frac{5}{13}$	$\frac{2}{13}$	$\boldsymbol{k}$

(a) Find the value of *k*. **1**

$$\frac{3}{13}+\frac{5}{13}+\frac{2}{13}+k=1$$

$$k=1-\left(\frac{3}{13}+\frac{5}{13}+\frac{2}{13}\right)=\frac{3}{13} \quad \boxed{\checkmark}$$

(b) Find E(*X*). **2**

$$E(X)=0\times\frac{3}{13}+1\times\frac{5}{13}+2\times\frac{2}{13}+3\times\frac{3}{13} \quad \boxed{\checkmark}$$

$$=\frac{18}{13} \quad \boxed{\checkmark}$$

Question 12 (2 marks)

A crystal growth furnace is designed to manufacture components for a space shuttle. **2**

For proper growth the temperature must by controlled accurately by adjusting the input power. Suppose the relationship is given by $T = 0.1w^2 + 2.155w + 20$, where T is the temperature in Celsius and w is the (positive) input power in watts.
How much power is needed to maintain the temperature at $200°C$?

$$200 = 0.1w^2 + 2.155w + 20$$

$$0.1w^2 + 2.155w - 180 = 0$$

$$w = \frac{-2.155 \pm \sqrt{2.155^2 - 4(0.1)(-180)}}{2 \times 0.1}$$ ☑

$$w = 32.9983, -54.5483$$

$$w = 32.9983, w \geq 0$$ ☑

Question 13 (4 marks)

For the arithmetic sequence 10, 4, -2 ….

(a) Calculate the value of the fourteenth term. **2**

$a = 10, d = -6 \quad n = 14$ ☑

$$T_{14} = 10 + (14 - 1) \times -6$$

$$= -68$$ ☑

(b) Find the sum of the first fourteen terms. **2**

$a = 10, L = -68, n = 14$

$$Sn = \frac{n}{2}(a + L)$$

$$= \frac{14}{2}(10 + -68)$$ ☑

$$= -406$$ ☑

Question 14 (2 marks)

Evaluate $\int_0^4 x\left(x^2+6\right)dx$ **2**

$$\int_0^4 x\left(x^2+6\right)dx=\int_0^4\left(x^3+6x\right)dx$$
$$=\left[\frac{x^4}{4}+3x^2\right]_0^4 \quad ☑$$
$$=\frac{4^4}{4}+3(4)^2-(0)$$
$$=112 \quad ☑$$

Question 15 (5 marks)

Nicky has opened a business to customise skateboards.

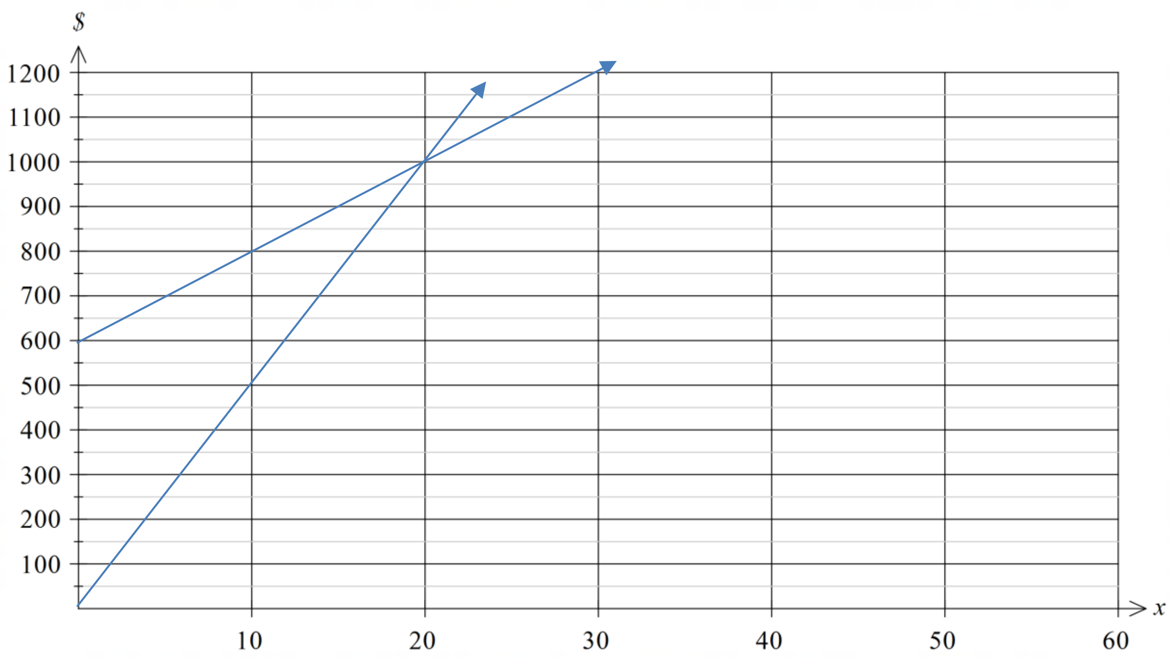
The cost of production is given by $C = 20x + 600$ dollars and income earned is given by $I = 50x$, where x is the number of skateboards produced.

(a) If the production cost is \$800, how many skateboards have been manufactured? **1**

$$800 = 20x + 600$$
$$20x = 200$$
$$x = 10 \quad \text{So 10 items were produced.} \quad ☑$$

(b) Graph both equations on the axes provided. **2**

☑☑



(c) What is the minimum number of skateboards that Nicky must sell to make a profit? **2**

To make a profit, income must exceed cost.

$50x > 600 + 20x$ ☑

$30x > 600$

$x > 20 \therefore x = 21.$ You must produce 21 items to make a profit. ☑

Question 16 (3 marks)

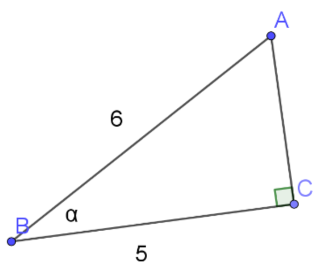
Given that $\cos\alpha = \frac{5}{6}$, and that $\tan\alpha < 0$ find the exact value of $\sin\alpha \times \sec\alpha$. **3**

Since $\cos\alpha > 0$ and $\tan\alpha < 0$, α is in the 4th quadrant.

$AC = \sqrt{6^2 - 5^2} = \sqrt{11}$ ☑

$\sin\alpha \times \sec\alpha = -\frac{\sqrt{11}}{6} \times \frac{6}{5}$ ☑

$= -\frac{\sqrt{11}}{5}$ ☑

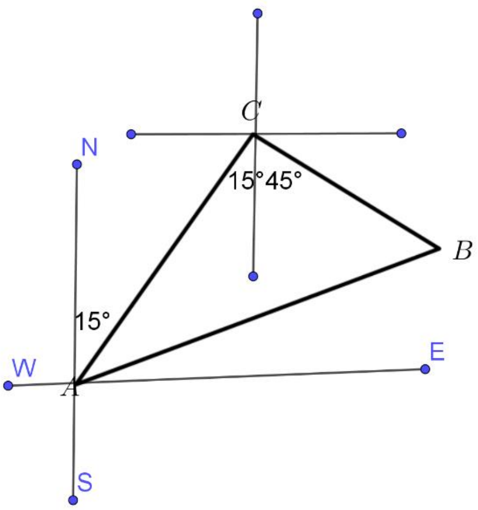


Question 17 (3 marks)

A runner leaves a point *A* and runs for 4 km on a bearing of $N15°E$.
She then runs 2 km *SW* from *C* to *B.*

(a) Use the diagram below to show why $\angle ACB = 60°$. **1**

☑



(b) Use the cosine rule to determine the exact distance from *A* to *B.* **2**

$AB^2 = 4^2 + 2^2 - 2 \times 4 \times 2 \cos 60°$ ☑

$AB^2 = 20 - 2 \times 4 \times 2 \times \frac{1}{2}$

$AB = \sqrt{12} = 2\sqrt{3}\,km$ ☑

Question 18 (3 marks)

In the city of Tangentia the average number of books in a residence is normally distributed **3**
Show that the probability at least one of them has between 4 and 13 books is 0.97 (2 dp)

$\text{Variance} = 9 \rightarrow \sigma = 3$ ☑

So a score between 4 and 13 books is from 2 standard deviations below the mean
to 1 standard deviation above. ☑

This means the probability of between 4 and 13 books will be 47.5% + 34% = 81.5%

P(at least one household with between 4 and 13 books) = $1 - 0.185^2 = 0.97(2\,dp)$ ☑

Question 19 (5marks)

(a) Use the fact that $\sec x = (\cos x)^{-1}$ to show that $\frac{d}{dx}(\sec x) = \sec x \tan x$ **2**

$$\frac{d}{dx}\sec x = \frac{d}{dx}(\cos x)^{-1}$$
$$= -(\cos x)^{-2} \times -\sin x \quad \boxed{\checkmark}$$
$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \sec x \tan x \boxed{\checkmark}$$

(b) Hence find $\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \, dx$ **3**

From (a) $\int \sec x \tan x \, dx = \sec x + C$

$$\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \, dx = \frac{1}{2}[\sec 2x]_0^{\frac{\pi}{6}} \quad \boxed{\checkmark}$$
$$= \frac{1}{2}\left[\sec\frac{\pi}{3} - \sec 0\right] \quad \boxed{\checkmark}$$
$$= \frac{1}{2}[2-1]$$
$$= \frac{1}{2} \quad \boxed{\checkmark}$$

Question 20 (6 marks)

(a) A function is defined by $f(x) = 2^{-x} + 1$. Complete the table of values below. **1**

x	0	1	2
$f(x)$	2	1.5	1.25

$\boxed{\checkmark}$

(b) Use the trapezoidal rule with all the above function values to estimate the value of $\int_0^2 (2^{-x}+1)\,dx$ correct to three decimal places. **2**

$$\int_0^2 (2^{-x}+1)\,dx \approx \frac{1}{2}(1)(2+2\times1.5+1.25)\times \quad \boxed{\checkmark}$$
$$\approx \frac{1}{2}6.25$$
$$\approx 3.125 \quad \boxed{\checkmark}$$

(c) Find the value of $\int_0^2 \left(2^{-x}+1\right)dx$ correct to three decimal places. **2**

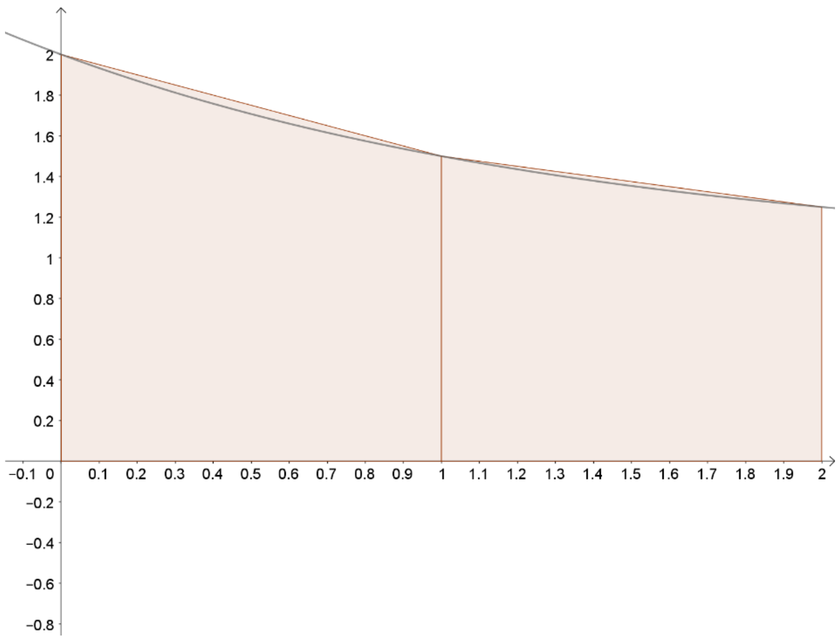
$$\int_0^2 \left(2^{-x}+1\right)dx = \left[\frac{2^{-x}}{-\ln 2}+x\right]_0^2$$

$$= \frac{2^{-2}}{-\ln 2}+2-\left(\frac{2^0}{-\ln 2}+1\right) \quad ✓$$

$$= 3.082\left(3\,dp\right) \quad ✓$$

(d) Why would you expect the value of the integral to be less than the approximation? **1**
(A diagram may prove useful.)

The curve is concave up. So the trapezium occupies more area than the area under the curve.



✓

Question 21 (8 marks)

A curve is defined by $y = x^3 - 3x^2 + 4$ over the domain $-2 \le x \le 3$.

(a) Find any turning points, identifying their nature. **3**

$y = x^3 - 3x^2 + 4$

$\frac{dy}{dx} = 3x^2 - 6x \left(\text{to find stat. pts let } \frac{dy}{dx} = 0 \right)$

$3x(x-2) = 0 \rightarrow x = 0, 2$ ☑

$x = 0, y = 4 \text{ and } x = 2, y = 0$

$\frac{d^2y}{dx^2} = 6x - 6$

$when\ x = 0, \frac{d^2y}{dx^2} = -6 < 0, \text{ concave down} \therefore (0,4) \text{ is a maximum t.p.}$ ☑

$when\ x = 2, \frac{d^2y}{dx^2} = 6 < 0, \text{ concave up} \therefore (2,4) \text{ is a minimum t.p.}$ ☑

(b) Find any points of inflection. **2**

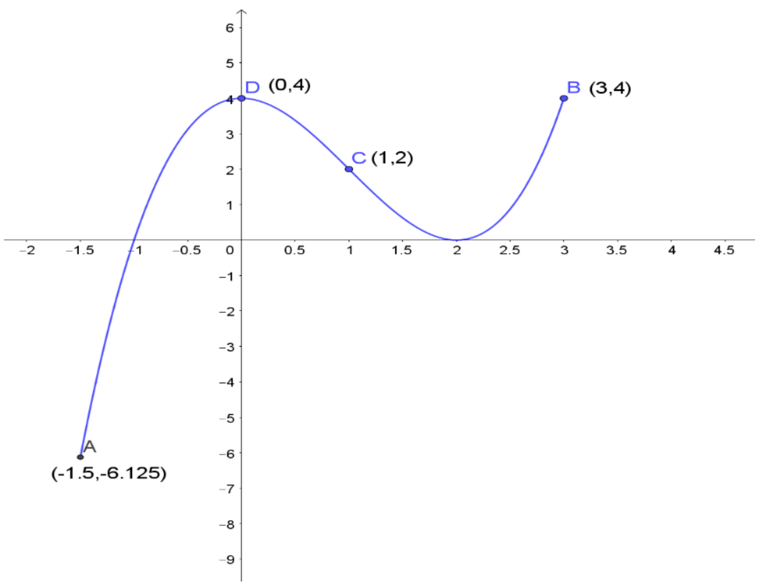
Let $\frac{d^2y}{dx^2} = 0$ to find points of inflection.

$6x - 6 = 0$

$x = 1, y = 2 \therefore (1, 2)$ is a possible point of inflection. ☑

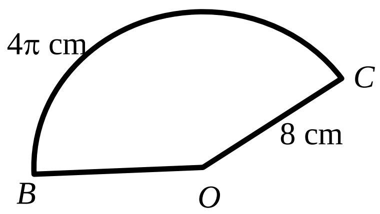
Since the curve changes concavity at $x = 0$ and $x = 2$ then (1,2) is a P.O.I ☑

(c) Draw a neat sketch of $y = x^3 - 3x^2 + 4$ over the domain $-1.5 \le x \le 3$, given the curve passes through (-1,0). Show intercepts, turning points, inflection(s) and endpoints. **3**



☑ ☑ ☑

Question 22 (2 marks)



NOT TO SCALE

OBC is a sector of a circle centre O and radius 8cm. The length of the arc *BC* is 4π cm. Calculate the exact area of the sector *OBC*. **2**

$$L = r\theta$$

$$4\pi = 8\theta$$

$$\theta = \frac{4\pi}{8} = \frac{\pi}{2} \quad \boxed{\checkmark}$$

$$A(\text{sector}) = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2} \times 8^2 \times \frac{\pi}{2}$$

$$= 16\pi \text{ cm}^2 \quad \boxed{\checkmark}$$

Question 23 (3 marks)

The time taken to complete a building project varies inversely with the number of builders. **3**
If the building can be completed in 180 days by 55 workers, find how many workers will be needed to finish the project in exactly 99 days

$$T \propto \frac{1}{W}$$

$$T = \frac{k}{W} \quad \boxed{\checkmark}$$

$$180 = \frac{k}{55}$$

$$k = 55 \times 180 = 9900 \quad \boxed{\checkmark}$$

$$99 = \frac{9900}{W}$$

$$W = 100 \quad \boxed{\checkmark}$$

So 100 workers will be required.

Question 24 (3 marks)

Solve $\ln(8-2x) = \frac{1}{2}\ln(4-x)$ **3**

$$\ln(8-2x) = \frac{1}{2}\ln(4-x)$$

$$\ln(8-2x) = \ln(4-x)^{\frac{1}{2}} \quad \boxed{\checkmark}$$

$$8-2x = \sqrt{4-x}$$

$$(8-2x)^2 = 4-x$$

$$4(4-x)^2 - (4-x) = 0$$

$$(4-x)(4(4-x)-1) = 0$$

$$(4-x)(15-4x) = 0$$

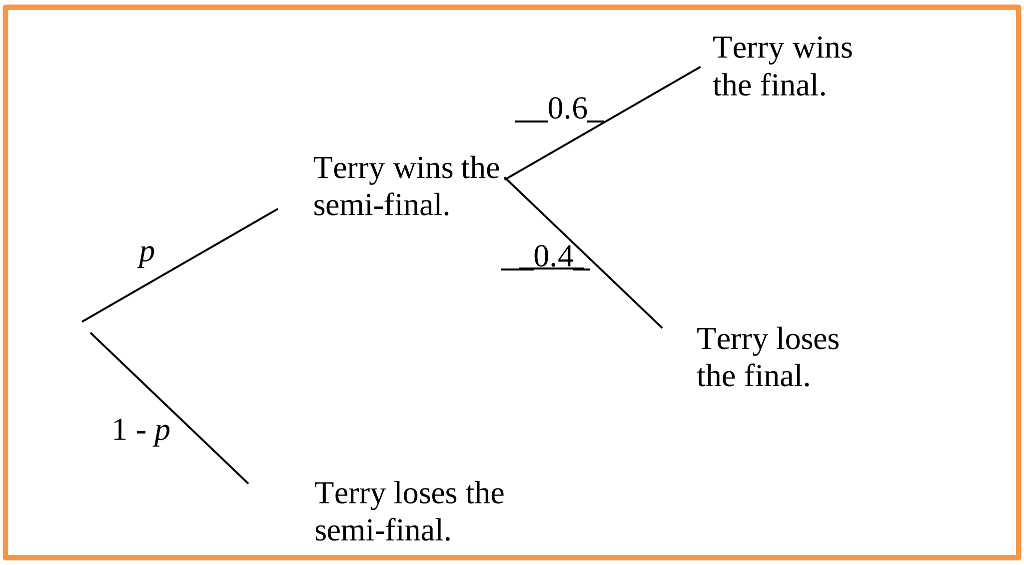
$$x = 4, \frac{15}{4} \quad \boxed{\checkmark}$$

$$x \neq 4, x < 4 \text{ so } x = \frac{15}{4} \quad \boxed{\checkmark}$$

Question 25 (5 marks)

Terry will play in the semi-final of a tennis tournament.
If Terry wins the semi-final she will progress to the final. If she loses she will not progress.
If Terry wins the final she will be champion.
The probability that Terry will win the semi-final is p.
If she wins the semi-final then the probability she will be the champion is 0.6

☑



(a) Complete the values in the tree diagram. **1**

(b) The probability that Terry will not be the champion is 0.58. Find the value of p. **2**

$$1 - p + p(0.4) = 0.58 \quad ☑$$
$$-0.6p = -0.42$$
$$p = 0.7 \quad ☑$$

(c) Given that Terry did not become the champion, find the probability she lost in the semi-final. **2**

$$P(E) = \frac{1-0.7}{0.58} \quad ☑$$
$$= \frac{0.3}{0.58}$$
$$= \frac{15}{29} \quad ☑$$

Question 26 (5 marks)

The half-life of a substance is the time taken to decay to half the original mass. Strontium-90 has a half-life of 28 days. Its mass S is given by $S = S_0 e^{-kt}$ where S is measured in mg and t is measured in days. A sample has an initial mass of 50 mg.

(a) Find the value of k correct to 3 decimal places. **2**

$S = S_0 e^{-kt}$

$25 = 50e^{-k28}$

$-28k = \ln(0.5)$ ☑

$k = \dfrac{\ln(0.5)}{-28} = \dfrac{\ln 2}{28} = 0.025\,(3\text{dp})$ ☑

(b) Use the exact value of k to find the mass remaining after 40 days. **1**

$S = S_0 e^{-kt}$

$S = 50e^{-40\frac{\ln 2}{28}}$

$S = 18.575\,\text{mg}\,(3\text{dp})$ ☑

(c) How long does it take (to the nearest day) for the sample to decay to a mass of 2 mg. **2**

$S = S_0 e^{-kt}$

$25 = 50e^{-k28}$

$-28k = \ln(0.5)$ ☑

$k = \dfrac{\ln(0.5)}{-28} = \dfrac{\ln 2}{28} = 0.025\,(3\text{dp})$ ☑

Question 27 (7 marks)

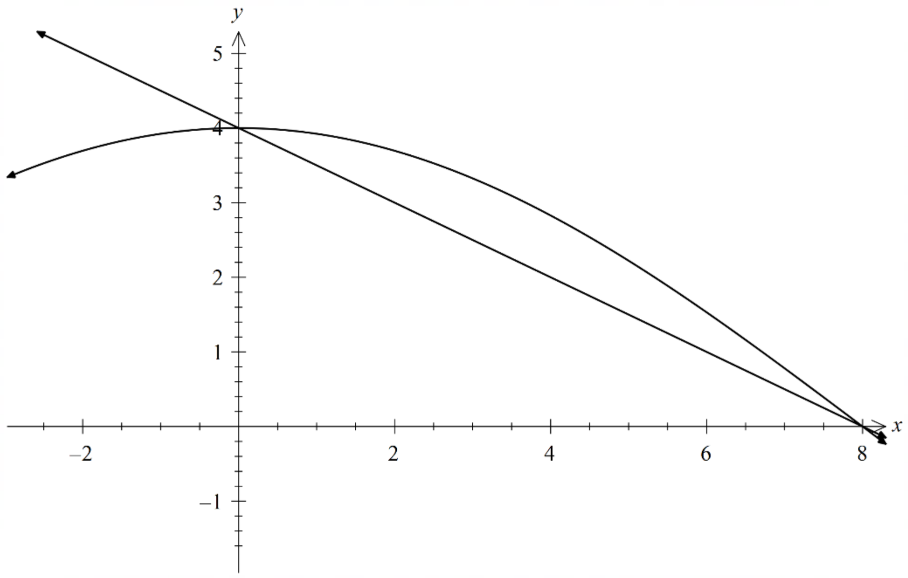
Two functions are given by $y = 4\cos\left(\dfrac{\pi}{16}x\right)$ and $y = -\dfrac{1}{2}x + 4$

(a) Show that the functions have a point of intersection at (8,0) **2**

$$y = 4\cos\left(\frac{8\pi}{16}\right) = 4\cos\frac{\pi}{2} = 0 \quad ☑$$

$$y = -\frac{1}{2}\times 8 + 4 = -4 + 4 = 0 \quad ☑$$

(b) Sketch both functions over the domain [0,8] **2**



☑ ☑

(c) Find the exact area enclosed by the curves over this domain. **3**

$$A = \int_0^8 \left(4\cos\frac{\pi}{16}x - \left(-\frac{1}{2}x + 4\right)\right)dx \quad ☑$$

$$= \left[\frac{64}{\pi}\sin\frac{\pi}{16}x + \frac{x^2}{4} - 4x\right]_0^8 \quad ☑$$

$$= \frac{64}{\pi}\times 1 + 16 - 32 - (0) = \left(\frac{64}{\pi} - 16\right)u^2 \quad ☑$$

Question 28 (9 marks)

(a) If $y = 1 - xe^{-x}$ find $\frac{dy}{dx}$. **3**

Let $u = -x$ and $v = e^{-x}$ $\rightarrow u' = -1$ and $v' = -e^{-x}$ ☑

$\frac{dy}{dx} = 0 + -e^{-x} + (-x)(-e^{-x})$ ☑

$= -e^{-x} + x(e^{-x})$ ☑

(b) Find the equations of the tangent and the normal to $y = 1 - xe^{-x}$ at the point $P(0,1)$ **4**

At $x = 0$ $m(\text{tangent}) = -1$ ☑

$y - 1 = -1(x - 0)$

$y = -x + 1$ ☑

At $x = 0$ $m(\text{normal}) = 1$ ☑

$y - 1 = 1(x - 0)$

$y = x + 1$ ☑

(c) If the tangent and normal meet the x-axis at A and B respectively, find the area of the triangle ABP. **2**

Vertices are at (1,0), (-1,0) and (0,1) ☑

$A = \frac{1}{2} \times 2 \times 1$

$= 1u^2$ ☑

Question 29 (6 marks)

Albert borrows \$200,000 at 6% per annum compounded monthly. He makes monthly payments of \$2000 at the end of each month. B_n is the balance owing after n months,

(a) Show that the amount owing after 2 years is given by $B_{24} = \$174568$ **3**

$r = 0.005$, $P = \$2000$ Let B_n be the balance owed after n months.

$$B_1 = 200000 \times 1.005 - 2000 \quad ☑$$

$$B_2 = (200000 \times 1.005 - 2000) \times 1.005 - 2000$$

$$= 200000 \times 1.005^2 - 2000 \times 1.005 - 2000.$$

$$= 200000 \times 1.005^2 - 2000(1 + 1.005)$$

$$B_{24} = 200000 \times 1.005^{24} - 2000(1 + 1.005 + 1.005^2 \ldots 1.005^{23}) \quad ☑$$

$$= 200000 \times 1.005^{24} - 2000\left(\frac{1.005^{24} - 1}{1.005 - 1}\right)$$

$$B_{24} = \$174568.04 \quad ☑$$

(b) Albert decides he wants to pay the balance of the loan off after a further 8 years. **3**

What should his new monthly payment be?

Let A_n be the balance after a further n months, M be the monthly payment.

$$A_1 = 174568 \times 1.005 - M \quad ☑$$

$$A_2 = (174568 \times 1.005 - M) \times 1.005 - M$$

$$= 174568 \times 1.005^2 - M(1 + 1.005)$$

$$A_{96} = 200000 \times 1.005^{96} - 2000(1 + 1.005 + 1.005^2 \ldots 1.005^{96}) \quad ☑$$

$$= 174568 \times 1.005^{96} - M\left(\frac{1.005^{96} - 1}{1.005 - 1}\right) = 0$$

$$-M\left(\frac{1.005^{96} - 1}{1.005 - 1}\right) = -174568 \times 1.005^{96}$$

$$M = \frac{174568 \times 1.005^{96}}{\left(\frac{1.005^{96} - 1}{1.005 - 1}\right)}$$

$$M = \$2294.07 \quad ☑$$

Question 30 (6 marks)

A probability density function is given by $f(x)=\begin{cases}\dfrac{1}{2\sqrt{4-x}} & ;0\le x\le 3\\ 0 & ;\text{otherwise}\end{cases}$

(a) Prove that this is a valid probability density function, (you may assume $f(x)\ge 0$). **2**

$$\int_0^3 \frac{1}{2\sqrt{4-x}}\,dx=\frac{1}{2}\int_0^3(4-x)^{-\frac{1}{2}}\,dx$$

$$=\frac{1}{2}\left[\frac{(4-x)^{\frac{1}{2}}}{-\frac{1}{2}}\right]_0^3 \quad ☑$$

$$=\left[-\sqrt{4-x}\right]_0^3$$

$$=-\left(\sqrt{1}-\sqrt{4}\right) \quad ☑$$

$$=1 \text{ as required}$$

(b) Find the cumulative distribution function. **2**

$$F(x)=\int_0^x \frac{1}{2\sqrt{4-x}}\,dx \quad ☑$$

$$=\frac{1}{2}\int_0^x(4-x)^{-\frac{1}{2}}\,dx$$

$$=\left[-\sqrt{4-x}\right]_0^x$$

$$=-\sqrt{4-x}-\left(-\sqrt{4-0}\right)$$

$$=2-\sqrt{4-x} \quad ☑$$

(c) Find the median of the probability density function. **2**

Median occurs when $F(x)=\frac{1}{2}$ ☑

$$2-\sqrt{4-x}=\frac{1}{2}$$

$$\sqrt{4-x}=\frac{3}{2}$$

$$4-x=\frac{9}{4}$$

$$x=\frac{7}{4} \quad ☑$$